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E.g. 2.3.7

Similar to Ex 2.3.6

Note that $a_{W,X,Y} \otimes id_Z = I_A \cdot \rho(g)(I_A) = I_A$.

E.g. 2.3.8

$\mathcal{C} = \mathcal{C}_G(A)$. Note that asso isom $a_{-,-,-}$ gives a mapping from $G \times G \times G$ to A, i.e. $a \in \mathrm{Fun}(G^3, A) = C^3$ (§1.7).

(View A as a trivial G-mod)

$$C^3 \xrightarrow{d^4} C^4$$

$$\omega \longmapsto d^4\omega, \quad d^4\omega(W,X,Y,Z) = [W \cdot \underset{\omega(X,Y,Z)}{\omega(X,Y,Z)}] \cdot \omega(WX,Y,Z)^{-1} \omega(W,XY,Z) \cdot \omega(W,X,YZ)^{-1} \cdot \omega(W,X,Y)$$

Define a new asso isom $a^{\omega}_{-,-,-}$ by

$$a^{\omega}_{\delta_g,\delta_h,\delta_m} = \omega(g,h,m) \cdot id_{\delta_{ghm}} : (\delta_g \otimes \delta_h) \otimes \delta_m \longrightarrow \delta_g \otimes (\delta_h \otimes \delta_m)$$

Then $d^4\omega = 0 \Rightarrow$ pentagon axiom holds.

"Linear Version"

View $\mathcal{C}_G(k^\times)$ as subset of Vec_G. Define $\mathcal{C}^{\omega}_G(k^\times)$ following above way, and extend the associativity isom of $\mathcal{C}^{\omega}_G$ to Vec^{ω}_G by additivity to arbitrary direct sum of objects δ_g.

In $\mathcal{C}_G(k^\times)$, by def of l, r,

$$1 \otimes l_{\delta_g} = l \otimes id_{\delta_g} \circ \omega(1,1,g)^{-1} \Rightarrow l_{\delta_g} = \omega(1,1,g)^{-1}$$

$$r_{\delta_h} \otimes 1 = id_{\delta_h} \otimes r \circ \omega(g,1,1) \Rightarrow r_{\delta_h} = \omega(g,1,1)$$

Therefore triangle axiom says $\omega(g,1,h) = \omega(g,1,1)\omega(1,1,h)$.

By $l_X = r_X = id_X$, $\omega(g,1,1) = \omega(1,1,g) = 1_A \Rightarrow \omega(g,1,h) = 1_A, \forall g,h \in G$.

A cocycle satisfying this condition is called to be normalized.

Rmk 2.3.10 We will show in Prop 2.6.1 that cohomologically equivalent ω's give rise to equivalent monoidal cat.

Rmk 2.3.12 Let $\mathcal{C}$ be a cat. Then the cat $\mathrm{End}(\mathcal{C})$ is a monoidal cat, where $\otimes$ is given by composition of functors, the asso isom is identity. The unit obj is $id_{\mathcal{C}}$.

If $\mathcal{C}$ is abelian, the cat of additive, left exact, right exact and exact endofunctors are monoidal.

Eg. 2.3.13 Let A be an associative ring with unit. $(A\text{-bimod}, \otimes_A, a, \underset{\text{regular mod}}{{}_A A_A}, \iota)$ is monoidal.

If A is commu, A-bimod has a full monoidal subcat A-mod, regarded as bimod in which left and right actions coincide.

If X is a scheme, $(QCoh(X), - \underset{\mathcal{O}_X}{\otimes} -, a, \mathcal{O}_X, \iota)$ is monoidal. If $X = \mathrm{Spec} A$, then $QCoh(X) = A\text{-mod}$.

Similar, if A is a f.d. alg. A-bimod, A-mod (of f.d.) are monoidal.

Examples from geometry

$$\begin{array}{c} Coh(X). \quad X: \text{Noether scheme} \\ \cup \\ VB(X) \end{array}$$

$Loc(X)$ of locally constant sheaves of f.d. k-vector spaces

Eg. 2.3.14 The cat of tangles.

Let $S_{m,n}$ be the disjoint union of m S^1 and n $I = [0,1]$.
A tangle is a smooth embedding $f: S_{m,n} \longrightarrow \underbrace{\mathbb{R}^2 \times [0,1]}_{\text{coordinate}: x, y, z}$ s.t. boundary maps to boundary and interior maps to interior. (braids $\amalg$ arcs $\amalg$ links)

inputs := points of f with $z = 0$
output := — — $z = 1$.

- inputs and output have vanishing y-coordinates.

$\widetilde{T}_{p,q} :=$ { tangles having p inputs and q outputs }

$T_{p,q} := \widetilde{T}_{p,q} / (\text{isotopy})$ (inputs and outputs preserving "$y=0$" when moved)

Define a composition map $T_{p,q} \times T_{q,r} \longrightarrow T_{p,r}$, induced by the concatenation of tangles. Namely, if $s \in T_{p,q}$ and $t \in T_{q,r}$, pick representatives $\tilde{s} \in \widetilde{T}_{p,q}$, $\tilde{t} \in \widetilde{T}_{q,r}$ such that the inputs of $\tilde{t}$ coincide with the outputs of $\tilde{s}$, concatenation them, perform an appropriate reparametrization, and rescale $z \to \frac{z}{2}$. The obtained tangle represents the desired composition ts. (independent from the choice of $\tilde{t}$ and $\tilde{s}$).

Define a monoidal cat $\mathcal{T}$ called the cat of tangles:

cat structure:

$\mathrm{ob}\,\mathcal{T} = \mathbb{Z}_{\geq 0}$.

$\mathrm{Hom}_{\mathcal{T}}(p, q) = T_{p,q}$. composition as above. $id_p \in T_{p,p}$, $\underset{p \geq 1}{}$ represented by trivial braid. $id_0 = \phi$.

monoidal structure

obj: $m \otimes n = m + n$

mor: $t_1 \otimes t_2$ represented by $\tilde{t}_1 \amalg \tilde{t}_2$ (in $\tilde{t}_1 \amalg \tilde{t}_2$, any points of $\tilde{t}_1$ has a smaller x-coordinate the $\tilde{t}_2$).

Exercise 2.3.15 Check the following:

(1) The tensor product $t_1 \otimes t_2$ is well-defined, and its def makes $\otimes$ a bifunctor.

choose $\tilde{t}_1 \xrightarrow{\text{isotopy}} \tilde{t}_1'$, $\tilde{t}_2 \xrightarrow{\text{isotopy}} \tilde{t}_2'$. Mae the whole space s.t. $\tilde{t}_1, \tilde{t}_1'$ in half-space $\{x<0\}$, $\tilde{t}_2, \tilde{t}_2'$ in $\{x>0\}$.
Then the isotopy $\tilde{t}_1 \to \tilde{t}_1'$ can be chosen to be contained in $\{x<0\}$.
- - - $\tilde{t}_2 \to \tilde{t}_2'$ — " — " — " — - - $\{x>0\}$.

choose $\tilde{t}_1 \xrightarrow{\text{isotopy}} t_1'$, $\tilde{t}_2 \xrightarrow{\text{isotopy}} t_2'$. Mae the whole space s.t. $\tilde{t}_1, \tilde{t}_1'$ in $\{x<0\}$, $\tilde{t}_2, \tilde{t}_2'$ in $\{x>0\}$.

Then the isotopy $\tilde{t}_1 \to \tilde{t}_1'$ can be chosen to be contained in $\{x<0\}$.

- - - $\tilde{t}_2 \to \tilde{t}_2'$ - - - - - - $\{x>0\}$.

obviously $\tilde{t}_1 \amalg \tilde{t}_2 \xrightarrow{\text{isotopy}} \tilde{t}_1 \amalg \tilde{t}_2'$.

Similar $t_1 t_1' \otimes t_2 t_2' = (t_1 \otimes t_2)(t_1' \otimes t_2')$ (compose t_1 and t_1' (resp t_2 and t_2') in half-space in $\{x<0\}$ (resp $\{x>0\}$)).

(2) There is an obvious associativity isom for $\otimes$.

$$
\begin{array}{l}
a_{p,q,r} : (p \otimes q) \otimes r \longrightarrow p \otimes (q \otimes r) \\
\quad \| \\
\mathrm{id}_{p+q+r} : p+q+r \longrightarrow p+q+r .
\end{array}
$$

§2.4 Monoidal functors and their morphisms.

Def 2.4.1 Let $(\mathcal{C}, \otimes, 1, a, u)$ and $(\tilde{\mathcal{C}}, \tilde{\otimes}, \tilde{1}, \tilde{a}, \tilde{u})$ be monoidal cat. A monoidal functor from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ is a pair (F, J), where $F : \mathcal{C} \to \tilde{\mathcal{C}}$ is a functor, and natural isom

$$J_{X,Y} : F(X) \tilde{\otimes} F(Y) \xrightarrow{\simeq} F(X \otimes Y).$$

such that $F(1) \simeq \tilde{1}$ and diagram

$$
\begin{tikzcd}
(FX \tilde{\otimes} FY) \tilde{\otimes} FZ \arrow[r, "\tilde{a}_{FX,FY,FZ}"] \arrow[d, "J_{X,Y} \tilde{\otimes} \mathrm{id}_{FZ}"'] \arrow[dr, phantom, "\circlearrowright"] & FX \tilde{\otimes} (FY \tilde{\otimes} FZ) \arrow[d, "\mathrm{id}_{FX} \tilde{\otimes} J_{Y,Z}"] \\
F(X \otimes Y) \tilde{\otimes} FZ \arrow[d, "J_{X \otimes Y, Z}"'] & FX \tilde{\otimes} F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(a_{X,Y,Z})"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}
$$

is commu. (monoidal structure axiom)

A monoidal functor F is said to be an equi of monoidal cat if it is an equiv of cat.

Rmk 2.4.2 A Functor can be equipped with different monoid structure or not admit any monoid structure.

※ (F, J) defined above also preserves l and r. i.e. $\exists\, \varphi : \tilde{1} \to F(1)$ isom s.t.

$$
\begin{tikzcd}
F(1 \otimes X) \arrow[r, "F(l_X)"] & F(X) \\
F(1) \tilde{\otimes} F(X) \arrow[u, "J"] \arrow[ur, phantom, "\circlearrowright"] & \\
\tilde{1} \tilde{\otimes} F(X) \arrow[u, "\varphi \tilde{\otimes} \mathrm{id}"] \arrow[uur, "\tilde{l}_{F(X)}"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(X \otimes 1) \arrow[r, "F(r_X)"] & F(X) \\
F(X) \tilde{\otimes} F(1) \arrow[u, "J"] \arrow[ur, phantom, "\circlearrowright"] & \\
F(X) \tilde{\otimes} \tilde{1} \arrow[u, "\mathrm{id} \tilde{\otimes} \varphi"] \arrow[uur] &
\end{tikzcd}
$$

commu for any X. $(*)$

In the case $X = 1$, diagram $(*)$ says

$$
\begin{tikzcd}
\tilde{1} \tilde{\otimes} F(1) \arrow[r, "\tilde{l}_{F(1)}"] \arrow[d, "\varphi \tilde{\otimes} \mathrm{id}_{F(1)}"'] \arrow[dr, phantom, "\circlearrowright"] & F(1) \arrow[d, "F(l_1)^{-1}"] \\
F(1) \tilde{\otimes} F(1) \arrow[r, "J_{1,1}"'] & F(1 \otimes 1)
\end{tikzcd}
$$

Since $\tilde{l}_{F(1)}, F(l_1)^{-1}, J_{1,1}$ are all isom, we obtain an isom $J_{1,1}^{-1} \circ F(l_1)^{-1} \circ \tilde{l}_{F(1)} : \tilde{1} \tilde{\otimes} F(1) \longrightarrow F(1) \tilde{\otimes} F(1)$.

The existence and uniqueness of φ is implied by "$- \tilde{\otimes} F(1)$ is fully-faithful". And it is easy to see "$- \tilde{\otimes} \tilde{1}$ is fully-faithful and $\tilde{1} \simeq F(1) \Rightarrow - \tilde{\otimes} F(1)$ is fully-faithful".

Prop 2.4.3: For any X, and φ defined above, diagrams

$$
\begin{tikzcd}
\tilde{1} \tilde{\otimes} F(X) \arrow[r, "\tilde{l}_{F(X)}"] \arrow[d, "\varphi \tilde{\otimes} \mathrm{id}_{FX}"'] \arrow[dr, phantom, "\circlearrowright"] & F(X) \arrow[d, "F(l_X)^{-1}"] \\
F(1) \tilde{\otimes} F(X) \arrow[r] & F(1 \otimes X)
\end{tikzcd}
\tag{2.25}
$$

and

$$
\begin{tikzcd}
F(X) \tilde{\otimes} \tilde{1} \arrow[r, "\tilde{r}_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)} \tilde{\otimes} \varphi"'] \arrow[dr, phantom, "\circlearrowright"] & F(X) \arrow[d, "F(r_X)^{-1}"] \\
F(X) \tilde{\otimes} F(1) \arrow[r, "J_{X,1}"] & F(X \otimes 1)
\end{tikzcd}
\tag{2.26}
$$

are commu

To prove this, we need to prove a lemma.

$$F(x)\tilde\otimes F(1) \xrightarrow{J_{x,1}} F(x\otimes 1)$$

are commu

To prove this, we need to prove a lemma:

<u>Lemma 1:</u> Let $\mathcal{C}$ be a cat, $F,G,H\in \mathrm{End}\,\mathcal{C}$, $f:F\to G$, $g:G\to H$, $h:F\to H$ are natural trans. $E:\mathcal{C}\to\mathcal{C}$ equiv of cat, E^{-1} a quasi-inverse of E. Then

$$\forall x\quad \begin{tikzcd} F(x) \arrow[r, "f_x"] \arrow[dr, "h_x"'] & G(x) \arrow[d, "g_x"] \\ & H(x) \end{tikzcd} \quad\Longleftrightarrow\quad \forall x,\ \begin{tikzcd} F(Ex) \arrow[r, "f_{Ex}"] \arrow[dr, "h_{Ex}"'] & G(Ex) \arrow[d, "g_{Ex}"] \\ & H(Ex) \end{tikzcd}$$

is commu. is commu. (*)

Pf: "$\Longrightarrow$" Obviously

"$\Longleftarrow$" Consider the following diagram

$$\begin{tikzcd} F(EE^{-1}(x)) \arrow[r, "f_{EE^{-1}(x)}"] \arrow[drr, "h_{EE^{-1}(x)}"] \arrow[dd, "F(\alpha_x)"] & G(EE^{-1}(x)) \arrow[dr, "g_{EE^{-1}(x)}"] \arrow[dd, "G(\alpha_x)"] & \\ & & H(EE^{-1}(x)) \arrow[d, "H(\alpha_x)"] \\ F(x) \arrow[r, "f_x"] \arrow[rr, bend right, "h_x"'] & G(x) \arrow[r, "g_x"] & H(x) \end{tikzcd}$$

where $\alpha: EE^{-1}\to \mathrm{Id}_{\mathcal{C}}$ natural isom. All rectangles commute by the naturality of f,g,h. The upper triangle commutes by condition. Therefore the lower triangle commutes. $\square$.

<u>Pf of Prop 2.4.3.</u> By lemma 1, to prove (2.25) is commu, only need to prove

$$\begin{tikzcd} \tilde 1\tilde\otimes F(1\otimes x) \arrow[r, "l_{F(1\otimes x)}"] \arrow[d, "\varphi\otimes id_{F(1\otimes x)}"'] \arrow[dr, phantom, "\circlearrowright"] & F(1\otimes x) \arrow[d, "F(l_{1\otimes x})^{-1}"] \\ F(1)\tilde\otimes F(1\otimes x) \arrow[r, "J_{1,1\otimes x}"'] & F(1\otimes(1\otimes x)) \end{tikzcd}$$

is commu. To prove this, it is sufficient to establish the commutativity of the following diagram:

$$\begin{tikzcd}[column sep=small] \tilde 1\tilde\otimes F(1\otimes x) \arrow[rrrrr, "l"] \arrow[ddddd, "\varphi\otimes id"'] & & & & & F(1\otimes x) \arrow[ddddd, "F(l)^{-1}"] \\ & \tilde 1\tilde\otimes(F(1)\otimes F(x)) \arrow[ul, "id\otimes J"'] \arrow[drr, "l"] \arrow[ddd, "\varphi\otimes id"] & & & & \\ & & (1\otimes F(1))\otimes F(x) \arrow[ul, "a"'] \arrow[r, "l\otimes id"] \arrow[d, "\varphi\otimes id"] & F(1)\otimes F(x) \arrow[uurr, "J"] \arrow[d, "F(l)^{-1}\otimes id"] & & \\ & & (F(1)\otimes F(1))\otimes F(x) \arrow[r, "J\otimes id"'] \arrow[dl, "a"] & F(1\otimes 1)\otimes F(x) \arrow[dr, "J"] & & \\ & F(1)\otimes(F(1)\otimes F(x)) \arrow[dl, "id\otimes J"] & & & F((1\otimes 1)\otimes x) \arrow[uuuur, "F(l\otimes id)"'] \arrow[dr, "a"] & \\ F(1)\tilde\otimes F(1\otimes x) \arrow[rrrrr, "J"] & & & & & F(1\otimes(1\otimes x)) \end{tikzcd}$$

$\circlearrowright$: by composition $\qquad$ $\circlearrowright$: by monoidal structure axiom.

$\circlearrowright$: by naturality

$\circlearrowright$: by def of l.

To prove (2.26), we prove that φ is canonical for r, i.e. we have

$$(\maltese)\qquad \begin{tikzcd} F(1)\tilde\otimes\tilde 1 \arrow[r, "\tilde r_{F(1)}"] \arrow[d, "id_{F(1)}\tilde\otimes\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & F(1) \arrow[d, "F(r_1)^{-1}=F(l_1)^{-1}"] \\ F(1)\tilde\otimes F(1) \arrow[r, "J_{1,1}"] & F(1\otimes 1) \end{tikzcd}.$$

Pf: Consider.

$$\begin{tikzcd} (F(1)\tilde\otimes F(1))\tilde\otimes F(1) \arrow[rr, "a_{F1,F1,F1}"] \arrow[d, "J\otimes id_{F(1)}"'] & & F(1)\tilde\otimes(F(1)\tilde\otimes F(1)) \arrow[d, "id_{F(1)}\otimes J"] \\ F(1\tilde\otimes 1)\tilde\otimes F(1) \arrow[r, "F(l_1)\tilde\otimes id"] \arrow[dd, "J"'] & F(1)\otimes F(1) \arrow[d, "J"] & F(1)\tilde\otimes F(1\otimes 1) \arrow[l, "id\tilde\otimes F(r_1)"'] \arrow[dd, "J"] \\ & F(1\otimes 1) & \\ F((1\otimes 1)\otimes 1) \arrow[rr, "F(a_{1,1,1})"'] \arrow[ur, "F(l_1\otimes id)"] & & F(1\otimes(1\otimes 1)) \arrow[ul, "F(id\otimes r_1)"'] \end{tikzcd}$$

$\circlearrowright$: by triangle axiom

$\circlearrowright$: by naturality of J

: by triangle axiom

: by naturality of J.

Then by monoidal structure axiom, we obtain: , i.e.

$$(*) \quad F(l_1)\tilde{\otimes} id_{F(1)} \circ J_{1,1} \tilde{\otimes} id_{F(1)} = id_{F(1)} \tilde{\otimes} F(r_1) \circ id_{F(1)} \tilde{\otimes} J_{1,1} \circ a .$$

Consider:

$$\begin{tikzcd}
(F(1)\tilde{\otimes}\tilde{1})\tilde{\otimes}F(1) \arrow[rrrr, "\tilde{r}_{F(1)}\otimes id"] \arrow[drr, "a"] \arrow[ddd, "(id\otimes\varphi)\tilde{\otimes}id"'] & & & & F(1)\tilde{\otimes}F(1) \arrow[ddd, "F(r_1)^{-1}\tilde{\otimes}id"] \\
& & F(1)\tilde{\otimes}(\tilde{1}\tilde{\otimes}F(1)) \arrow[urr, "id\tilde{\otimes}\tilde{l}"] \arrow[d, "id\tilde{\otimes}(\varphi\tilde{\otimes}id)"] & & \\
& & F(1)\tilde{\otimes}(F(1)\tilde{\otimes}F(1)) \arrow[drr, dashed, "(J\otimes id)\circ a^{-1}"] & & \\
(F(1)\tilde{\otimes}F(1))\tilde{\otimes}F(1) \arrow[urr, "a"] \arrow[rrrr, "J\otimes id"'] & & & & F(1\otimes 1)\tilde{\otimes}F(1)
\end{tikzcd}$$

: by naturality of a

: by triangle axiom

: by $(*)$, note that $r_1 = l_1$ and $\overbrace{(\varphi^{-1}\tilde{\otimes} id_{F1})}^{id_{F1}\tilde{\otimes}} \circ id_{F1}\tilde{\otimes}\tilde{l}_{F(1)} = id_{F1}\tilde{\otimes}F(l_1)\circ id_{F1}\tilde{\otimes}J_{1,1}$ (by def of φ).

Since $-\otimes F(1)$ is fully-faithful, we have $(*)$. □

Similar to (2.25), one can prove (2.26).

<u>Def 2.4.5</u>: (F, J, φ) is a traditional def of a monoidal functor.

<u>Rmk 2.4.6</u>: One can safely identify $\tilde{1}$ with $F(1)$ using φ, and assume that $F(1)=\tilde{1}$ and $\varphi = id_{\tilde{1}}$. (Similarly to how we have identified $1\otimes X$ and $X\otimes 1$ with X and assumed that $l_X = r_X = id_X$.

<u>Rmk 2.4.7</u>: It is clear that the composition of monoidal functors is a monoidal functor. Also the identity functor has a natural structure of a monoidal functor.

<u>Def 2.4.8</u>: Let $(\mathcal{C}, \otimes, 1, a, l)$ and $(\tilde{\mathcal{C}}, \tilde{\otimes}, \tilde{1}, \tilde{a}, \tilde{l})$ be two monoidal cat, and let (F^1, J^1) and (F^2, J^2) be two monoidal functors from $\mathcal{C}$ to $\tilde{\mathcal{C}}$. A morphism (or a natural trans) of monoidal functors $\eta: (F^1, J^1) \longrightarrow (F^2, J^2)$ is a natural trans $\eta: F^1 \to F^2$ s.t. η_1 is an isom and $\forall X, Y \in \mathcal{C}$,

$$\begin{tikzcd}
F^1(X)\tilde{\otimes}F^1(Y) \arrow[r, "J^1_{X,Y}"] \arrow[d, "\eta_X\tilde{\otimes}\eta_Y"'] & F^1(X\otimes Y) \arrow[d, "\eta_{X\otimes Y}"] \\
F^2(X)\tilde{\otimes}F^2(Y) \arrow[r, "J^2_{X,Y}"] & F^2(X\otimes Y)
\end{tikzcd}$$

is commu.

Monoidal functors between two monoidal cat form a cat (with mor defined above).

<u>Rmk 2.4.9</u> If $\varphi_i : \tilde{1} \xrightarrow{\sim} F^i(1)$, $i=1,2$, then $\eta_1\circ\varphi_1 = \varphi_2$. So if one makes the convention that $\varphi_i = id_{\tilde{1}}$, one has $\eta_1 = id_{\tilde{1}}$.

Pf: Consider

$$\begin{tikzcd}
\tilde{1}\tilde{\otimes}F^2(1) \arrow[rrrr, "\tilde{l}"] \arrow[ddd, "(\eta_1\circ\varphi_1)\otimes id_{F^2(1)}"'] & & & & F^2(1) \arrow[ddd, "F(l_1)^{-1}"] \\
& \tilde{1}\tilde{\otimes}F^1(1) \arrow[ul, "id\tilde{\otimes}\eta_1"'] \arrow[rr, "\tilde{l}"] \arrow[d, "\varphi_1\otimes id"] & & F^1(1) \arrow[ur, "\eta_1"'] \arrow[d, "F(l_1)^{-1}"] & \\
& F^1(1)\tilde{\otimes}F^1(1) \arrow[rr, "J^1"'] \arrow[dl, "\eta_1\otimes\eta_1"'] & & F^1(1\otimes 1) \arrow[dr, "\eta_{1\otimes 1}"] & \\
F^2(1)\tilde{\otimes}F^2(1) \arrow[rrrr, "J^2"] & & & & F^2(1\otimes 1)
\end{tikzcd}$$

: by composition

: by naturality of η

$\circlearrowright$: by composition

$\circlearrowright$: by naturality of η

$\circlearrowright$: by naturality $\tilde{\ell}$

By the uniqueness of φ_2, we obtain $\varphi_2 = \eta_1 \circ \varphi_1$. $\square$

Rmk 2.4.10. If a monoidal functor is an equiv of monoidal cat, then it has a monoidal quasi-inverse.

Thus the monoidal autoequiv of any monoidal cat up to isom form a group w.r.t composition.

§2.5 Examples of monoidal functors

E.g. 2.5.1 forgetful functors:

$$\mathrm{Rep}\,G \longrightarrow \mathrm{Vec}$$
$$\mathrm{Rep}\,G \longrightarrow \mathrm{Rep}\,H, \quad H < G$$
$$\mathrm{Rep}\,G \longrightarrow \mathrm{Rep}\,H, \quad H \xrightarrow{f} G \text{ homomor}$$

E.g. 2.5.2: $f: H \to G$ homomor, define

$$f_*: \mathrm{Vec}_H \longrightarrow \mathrm{Vec}_G$$
$$\bigoplus_{h\in H} V_h \longmapsto \bigoplus_{g\in G} \bigoplus_{\substack{h\in H\\ f(h)=g}} V_h,$$

then f_* is monoidal. If $G = \{e\}$, then f_* is just the forgetful functor $\mathrm{Vec}_H \to \mathrm{Vec}$.

E.g. 2.5.3 Let k be a field, let A be a k-alg with unit, and let $\mathcal{C} = A\text{-mod}$ be the cat of left A-modules. then we have a functor

$$(2.29) \qquad F: A\text{-bimod} \longrightarrow \mathrm{End}\,\mathcal{C}$$
$$M \longmapsto M \otimes_A -$$

The functor is naturally monoidal. A similar functor $F: A\text{-bimod}^{(f.d.)} \longrightarrow \mathrm{End}\,\mathcal{C}$ can be defined if A is of f.d. and $\mathcal{C} = A\text{-mod}^{(f.d.)}$.

Prop 2.5.4 The functor (2.29) take values in the full monoidal subcat $\mathrm{End}_{re}(\mathcal{C})$ of right exact endfunctors of $\mathcal{C}$, and defines an equiv between the monoidal cat $A\text{-bimod}^{(f.d.)}$ and $\mathrm{End}_{re}(\mathcal{C})$.

Pf: The first statement is clear, since the tensor product functor is right exact.

To prove the second statement, let us construct the quasi-inverse F^{-1}. Define

$$F^{-1}: \mathrm{End}_{re}\,\mathcal{C} \longrightarrow A\text{-bimod}$$
$$G \longmapsto G(A),$$

this is clearly an A-bimod, since it is a left A-mod with a commuting action of $\mathrm{End}_A(A) = A^{op}$.

Obviously, $F^{-1} \circ F = - \otimes_A A \simeq \mathrm{id}_{A\text{-bimod}^{(f.d.)}}$. For any G, $F \circ F^{-1}(G) = G(A) \otimes_A -$, by Prop 1.8.10, $\exists\, V \in A\text{-bimod}^{(f.d.)}$ s.t. $G(-) = V \otimes_A -$, then $F \circ F^{-1}(G) = (V \otimes_A A) \otimes -$. By the canonical isom $V \otimes_A A \longrightarrow V$, one has $F \circ F^{-1} \simeq \mathrm{id}_{\mathrm{End}_{re}(\mathcal{C})}$. $\square$

Rmk 2.5.5. A similar statement is valid without the f.d. assumption, if one adds the condition that the right exact functors must commu with inductive limits.

E.g. 2.5.6 Let S be a monoid, let $\mathcal{C} = \mathrm{Vec}_S$ (similar to E.g. 2.3.6). Let us view $\mathrm{id}_{\mathcal{C}}$ as a monoidal functor. Let $\eta: \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ be a mor of monoidal functor (Def 2.4.8).

By def, we have

$$\begin{tikzcd}
\delta_g \otimes \delta_h \arrow[r, "="] \arrow[d, "\eta_g \otimes \eta_h"'] \arrow[dr, phantom, "\circlearrowright"] & \delta_{gh} \arrow[d, "\eta_{gh}"] \\
\delta_g \otimes \delta_h \arrow[r, "="] & \delta_{gh}
\end{tikzcd} \qquad \eta_g := \eta_{\delta_g}$$

$$\begin{tikzcd} {} \arrow[d, "\eta_g \otimes \eta_h"] \arrow[r, phantom, "\circlearrowright"] & {} \arrow[d, "\eta_{gh}"] \\ \delta_g \otimes \delta_h \arrow[r, "\cong"] & \delta_{gh} \end{tikzcd} \qquad \eta_g := \eta_{\delta_g}$$

$\Rightarrow$ $\eta_{gh} = \eta_g \cdot \eta_h$. Where $\{\delta_g\} = \mathcal{C}_S(k) \hookrightarrow \mathrm{Vec}_S$. Hence we can view η as a homomor of monoid from S to k. Conversely, if we have a homomor $\eta : S \to k$, we can define a mor of monoidal functor $id_{\mathcal{C}} \to id_{\mathcal{C}}$ via $\eta_{\delta_g} := \eta(g)$, $\forall g \in \mathcal{C}_S(k)$, and then extend to Vec_S by direct sum. Therefore morphisms $\eta : id_{\mathcal{C}} \to id_{\mathcal{C}}$ correspond to homomor of monoids $\eta : S \to k$.